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# Applicability of Heavy Quark Effective Theory to the Radiative Decay $B \rightarrow K^* \gamma$

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## Abstract

In the context of the observed decay  $B \rightarrow K^* \gamma$  the applicability of the Heavy Quark Effective Theory (HQET), treating both b and s as heavy quarks, is examined. We show that the heavy s-quark approximation, as can be found in the literature, is not reliable.

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# I. INTRODUCTION

The observed [1] rare decay  $B \rightarrow K^*\gamma$  (proceeding, as it does, through a flavor changing neutral current and absent at the tree level in the Standard Model(SM)), has attracted considerable attention, as it enables the testing of loop effects [2] involving the CKM matrix elements  $V_{tb}$  and  $V_{ts}$ ; furthermore additional contributions in the loop stemming from new bosons and fermions present in most of the extensions of the SM, holds forth the possibility that this process provides a window to new physics effects. However, the usual problem of relating hadron properties to quark-level theoretical inputs has to be addressed. While this, in principle, involves the notorious and as yet rather intractable features of non-perturbative QCD, approaches invoking effective Lagrangians could provide valuable estimates. The Heavy Quark Effective Theory (HQET) [3] is expected to be useful in this regard in so far as the b quark is concerned. However, the s quark in the final hadron can neither be considered heavy enough to enable the use of HQET nor sufficiently light to permit the exploitation of the chiral perturbation theory in an unambiguous manner. Nevertheless, attempts [4] have been made to apply HQET to both b and s quarks. The object of this paper is to examine the reliability of such an approach in the context of the decay in question, particularly because of the physics that lie in the loop and the fact that it is a rare decay which has been observed.

The paper is organised as follows: the next section sets forth the basic formalism. In section III we analyse the results using the Isgur, Scora, Grinstein and Wise (ISGW) model [5] (modified by Amundson [6]) and the Bauer, Stech and Wirbel (BSW) model [7] for the extraction of corrections of order  $\Lambda_{QCD}/m_s$  to results based on the consideration that both b and s

quarks are heavy, in order to arrive at an assessment as to whether such an approach is reliable. Section IV concludes the paper.

## II. FORMALISM

The exclusive decay  $B \rightarrow K^* \gamma$  is expected to be well described by the quark level process  $b \rightarrow s \gamma$  with reasonably small corrections from hadronic effects unlike what obtains for the decays of the K-meson, where nonperturbative long distance QCD contributions are expected to be substantial. Accordingly, integrating out the top-quark and W-boson fields, one arrives at an effective Hamiltonian for  $B$ -meson decays which comprises of a sum of ten operators [8, 9].

$$H_{eff} = -\frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \sum_{i=1}^{10} C_i(\mu) O_i(\mu) \quad (1)$$

where  $C_i(\mu)$ , the Wilson coefficients, arise from the renormalisation group equations to provide the scaling down to a subtraction point <sup>2</sup> appropriate to the problem viz.  $\mu \approx m_b$ . Of the ten operators  $O_i$  the one that contributes to  $B \rightarrow K^* \gamma$  is

$$O_7 = \frac{e}{32\pi^2} F_{\mu\nu} [m_b \bar{s} \sigma^{\mu\nu} (1 + \gamma_5) b + m_s \bar{s} \sigma^{\mu\nu} (1 - \gamma_5) b] \quad (2)$$

where  $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$  and  $F_{\mu\nu}$  is the electromagnetic field tensor. The relevant Wilson coefficient [8] is given by

$$\begin{aligned} C_7(\mu) = & \left[ \frac{\alpha_s(M_W)}{\alpha_s(m_b)} \right]^{16/23} \left\{ C_7(M_W) - \frac{8}{3} C_8(M_W) \left[ 1 - \left( \frac{\alpha_s(m_b)}{\alpha_s(M_W)} \right)^{2/23} \right] \right. \\ & \left. + \frac{232}{513} \left[ 1 - \left( \frac{\alpha_s(m_b)}{\alpha_s(M_W)} \right)^{19/23} \right] \right\} \end{aligned} \quad (3)$$

with

$$C_7(m_W) = -\frac{x}{2} \left[ \frac{\frac{2}{3}x^2 + \frac{5}{12}x - \frac{7}{12}}{(x-1)^3} - \frac{(\frac{3}{2}x^2 - x) \ln x}{(x-1)^4} \right] \quad (4)$$

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<sup>2</sup>Whether the subtraction point is to be taken at  $\mu = m_b$  is a point of debate. The popular range is from  $m_b/2$  to  $2m_b$ .

and

$$C_8(m_W) = -\frac{x}{4} \left[ \frac{\frac{1}{2}x^2 - \frac{5}{2}x - 1}{(x-1)^3} + \frac{3x \ln x}{(x-1)^4} \right] \quad (5)$$

where  $x = m_t^2/m_W^2$ , and  $C_8$  is the Wilson coefficient accompanying the operator  $O_8$ , which is

$$O_8 = \frac{\alpha}{4\pi} (\bar{s} \gamma_\mu \frac{1}{2} (1 - \gamma_5) b) (\bar{e} \gamma_\mu \gamma_5 e) \quad (6)$$

Even though one may later treat both  $b$  and  $s$  as heavy in the context of strong interaction effects, nevertheless taking cognisance of the fact that  $m_b \gg m_s$ , only the term involving  $m_b$  in the operator  $O_7$  need be retained. Thus the matrix element of interest becomes

$$\langle K^* \gamma | O_7 | \bar{B} \rangle = \frac{e}{32\pi^2} q_\mu \eta_\nu \langle K^* | \bar{s} [\gamma^\mu, \gamma^\nu] (1 + \gamma_5) b | \bar{B} \rangle m_b \quad (7)$$

where  $q_\mu$  is the four momentum of the photon and  $\eta_\mu$  its polarisation vector. It should be noted that here the matrix element of a tensor current between hadronic states is involved for which not much information is available. Heavy quark effective theory (HQET), however, comes to the rescue as the associated heavy-quark spin symmetry enables one to express the matrix element of the tensor operator in terms of the vector and axial vector form factors, which also occur in semi-leptonic decays and may be estimated in different phenomenological models. This is brought to bear upon the problem by first noticing that one may write the four-momentum carried away by the photon  $q = p - p'$  ( $p$  and  $p'$  being the four-momenta of the initial and final mesons) as  $q = M_B v - M_{K^*} v'$  where  $v$  and  $v'$  are the four-velocities of the initial and final mesons respectively. Furthermore, if both the bottom and the strange-quarks are taken to be heavy, the four-velocities of these quarks may, in the leading order, be approximated to be that of the corresponding

mesons, and accordingly, the ensuing relations

$$\not{v} b = b \quad (8)$$

$$\not{v}' s = s \quad (9)$$

permits one to write

$$\begin{aligned} \langle K^* \gamma | O_7 | \bar{B} \rangle &= \frac{e}{16\pi^2} m_b \eta_\mu [M_{K^*} \langle K^* | \bar{s} \gamma^\mu (1 + \gamma_5) b | \bar{B} \rangle \\ &+ M_B \langle K^* | \bar{s} \{ \gamma^\mu - 2v^\mu \} (1 - \gamma_5) b | B \rangle] \end{aligned} \quad (10)$$

In the rest frame of the  $B$ -meson [ $v^\mu = (1, \vec{0})$ ] and we have  $v \cdot \eta = 0$  (from the transversality of the photon). Thus the term in eq.7 containing  $v^\mu$  is unable to contribute. Therefore, in the limit of large  $b$  and  $s$ -quark masses the hadronic matrix elements of the tensor current get expressed in terms of the vector and axial vector form-factors (functions of  $y \equiv v \cdot v'$ ), which in turn admit the following general decomposition consistent with Lorentz invariance [10], to wit,

$$\langle K^*(v') | V_\mu^{sb} | \bar{B}(v) \rangle = i \sqrt{M_B M_{K^*}} \xi_V(y) \epsilon_{\mu\nu\alpha\beta} \epsilon^\nu v'^\alpha v^\beta \quad (11)$$

$$\begin{aligned} \langle K^*(v') | A_\mu^{sb} | \bar{B}(v) \rangle &= \sqrt{M_B M_{K^*}} \{ \xi_{A1}(y) (y + 1) \epsilon_\mu^* - \xi_{A2}(y) \epsilon^* \cdot v v_\mu \\ &- \xi_{A3}(y) \epsilon^* \cdot v v'_\mu \} \end{aligned} \quad (12)$$

where  $\epsilon$  is the polarisation vector of  $K^*$ ,  $V_\mu^{sb} = \bar{s} \gamma_\mu b$  and  $A_\mu^{sb} = \bar{s} \gamma_\mu \gamma_5 b$ . The factor  $\sqrt{M_B M_{K^*}}$  represents the change of normalisation of the states from the usual covariant one to that appropriate for HQET. From the underlying kinematics  $y \equiv v \cdot v' = \frac{M_B^2 + M_{K^*}^2}{2M_B M_{K^*}}$ . In the heavy quark limit it is very convenient to use a matrix representation of the mesonic states [3] namely  $B = -\frac{1+\not{v}}{2} \gamma_5$  and  $K^* = \frac{1+\not{v}'}{2} \not{\epsilon}$ , whereupon the matrix elements of the vector and axial

vector currents may be written as

$$\langle K^*(v') | V_\mu^{sb} | \bar{B}(v) \rangle = i\sqrt{M_B M_K^*} \xi(y) \text{Tr} \left\{ \not{\epsilon} \frac{(1 + \not{v}')}{2} \gamma_\mu \frac{(1 + \not{v})}{2} \gamma_5 \right\} \quad (13)$$

$$\langle K^*(v') | A_\mu^{sb} | \bar{B}(v) \rangle = \sqrt{M_B M_K^*} \xi(y) \text{Tr} \left\{ \not{\epsilon} \frac{(1 + \not{v}')}{2} \gamma_\mu \gamma_5 \frac{(1 + \not{v})}{2} \gamma_5 \right\} \quad (14)$$

the functions  $\xi_i(y)$  of eqs.(11) and (12) get related to a single universal form factor,  $\xi(y)$ , the Isgur-Wise function, through

$$\xi_i(y) = \alpha_i \xi(y) \quad (15)$$

with  $\alpha_V = \alpha_{A1} = \alpha_{A3} = 1$  and  $\alpha_{A2} = 0$ . The Isgur-Wise function satisfies the condition  $\xi(1) = 1$  at zero recoil ( $y = 1$ ). Thus in leading order the decay width is given by

$$\begin{aligned} \Gamma(B \rightarrow K^* \gamma) &= \frac{G_F^2}{64\pi^4} \alpha |V_{tb}|^2 |V_{ts}|^2 |C_7(\mu)|^2 m_b^2 \\ &\quad \left(1 - \frac{M_{K^*}^2}{M_B^2}\right) M_{K^*} |\xi(y)|^2 (y + 1) \\ &\quad [(M_B - M_{K^*})^2 (y + 2) + (M_B + M_{K^*})^2 (y - 1)] \quad (16) \end{aligned}$$

Whereas the b-quark may safely be considered to be heavy with respect to the QCD energy scale, this is at best an uncertain assumption in the case of the strange quark, though this has been the basis of several estimates [4]. To test the efficacy of such calculations it is important to study the magnitude of the  $\frac{1}{m_s}$  correction. This correction originates from two sources, namely, from the current on the one hand and the mesonic states on the other.

Consider the current  $\bar{h}_b \Gamma u_s$  (with  $\Gamma = \gamma_\alpha$  or  $\gamma_\alpha \gamma_5$ ). For the  $b$ -quark one can with impunity use the HQET effective field satisfying the condition  $\not{v} h_b = h_b$ . As we wish to calculate the non-leading corrections [11] in the case of the strange quark, we begin by recognising that its momentum is  $m_s v' + l$

(where  $l$  is a measure of the off-shellness of that quark in the daughter meson) and hence one has the equation of motion

$$(m_s \psi'' + \not{l} - m_s)u_s = 0 \quad (17)$$

which yields  $(1 - \psi'')u_s = \frac{\not{l}}{m_s}u_s$ , while in the limit of a heavy  $s$ -quark one has  $(1 - \psi'')u_s = 0$ . Thus the relevant correction arising from the current is given by

$$\bar{h}_b \Gamma u_s = \bar{h}_b \Gamma \left( \frac{1 + \psi''}{2} + \frac{1 - \psi''}{2} \right) u_s = \bar{h}_b \Gamma \left( 1 + \frac{\not{l}}{2m_s} \right) h_s + O\left(\frac{1}{m_s^2}\right) \quad (18)$$

Implementing this correction in the matrix element of the vector and axial vector currents between the relevant hadronic states and using the trace formalism for the mesons we arrive at the structure of the  $1/m_s$  correction emanating from the current

$$-\frac{1}{2m_s} \text{Tr}[\xi^\alpha \not{\epsilon}^* \left( \frac{1 + \psi''}{2} \right) \gamma_\alpha \Gamma \left( \frac{1 + \psi''}{2} \right) \gamma_5] \quad (19)$$

where  $\xi^\alpha$ , representing the expectation of  $l^\alpha$ , has the generic form

$$\xi^\alpha = \xi_+(v + v')^\alpha + \xi_-(v - v')^\alpha + \xi_3 \gamma^\alpha \quad (20)$$

However, the three functions so introduced are not independent. Indeed using the equation of motion  $iv.Dh_b = 0$  ( $D$  being the covariant derivative), the  $\xi'_i$ 's appearing here get related to each other through

$$\xi_3 = -(1 + y)\xi_+ - (1 - y)\xi_- \quad (21)$$

Again, the relation  $\partial_\mu(\bar{h}_s \Gamma h_b) = \bar{h}_s \overleftarrow{D}_\mu \Gamma h_b + \bar{h}_s \overrightarrow{D}_\mu \Gamma h_b$  when sandwiched between meson states yield the condition

$$\xi_- = \frac{\bar{\Lambda}}{2} \xi \quad (22)$$

where  $\bar{\Lambda}$  is the mass of the light degrees of freedom including the binding energy. Thus the  $\frac{1}{m_s}$  correction arising from the modification of the current introduces only one new function  $\xi_+$ .

As for the corrections arising from the meson states [3], it may be observed that  $|M(v)\rangle$  is an eigenstate of the leading order effective Lagrangian of HQET,

$$\mathcal{L} = \bar{h}_s i v \cdot D h_s \quad (23)$$

While, the physical mesonic state  $|M(p)\rangle$  is an eigenstate of full QCD. Expanding the Lagrangian in powers of  $\frac{1}{m_s}$  we may write

$$\mathcal{L} = \mathcal{L}_{HQET} + \frac{1}{2m_s} \mathcal{L}_1 + \frac{1}{4m_s^2} \mathcal{L}_2 + \dots \quad (24)$$

with

$$\begin{aligned} \mathcal{L}_1 &= \bar{h}_s (iD)^2 h_s + \frac{g}{2} \bar{h}_s \sigma^{\mu\nu} G_{\mu\nu} h_s \\ &= \mathcal{O}_1 + \mathcal{O}_2 \end{aligned} \quad (25)$$

Proceeding perturbatively the corrections to order  $\frac{1}{m_s}$  is given by

$$\frac{1}{2m_s} \langle K^* | i \int dy T \{ \bar{h}_s \gamma_\mu h_b(0), \mathcal{L}_1(y) \} | B \rangle \quad (26)$$

from which the term containing  $\mathcal{O}_1$ , namely

$$- \frac{\bar{\Lambda}}{2m_s} \{ \psi_1(y) Tr [ \not{\epsilon} \frac{(1 + \not{v})}{2} \Gamma \frac{(1 + \not{v})}{2} \gamma_5 ] \} \quad (27)$$

can be estimated, while that pertaining to  $\mathcal{O}_2$  is

$$\begin{aligned} & - \frac{\bar{\Lambda}}{2m_s} \{ i \psi_2(y) Tr [ v_\mu \gamma_\nu \not{\epsilon} \frac{(1 + \not{v})}{2} \sigma^{\mu\nu} \frac{(1 + \not{v})}{2} \Gamma \frac{(1 + \not{v})}{2} \gamma_5 ] \\ & + \psi_3(y) Tr [ \sigma_{\mu\nu} \not{\epsilon} \frac{(1 + \not{v})}{2} \sigma^{\mu\nu} \frac{(1 + \not{v})}{2} \Gamma \frac{(1 + \not{v})}{2} \gamma_5 ] \} \end{aligned} \quad (28)$$

Thus three new functions  $\psi_1$ ,  $\psi_2$  and  $\psi_3$  make their appearance from the  $\frac{1}{m_s}$  correction to the meson states. From Luke's theorem, which states that the

$\frac{1}{m_s}$  correction must vanish at the point  $y = 1$ , one arrives at the constraints

$$\psi_1(y = 1) = \psi_3(y = 1) = 0 \quad (29)$$

The four form-factors arising in the next-to-leading order approximation viz.  $\xi_+(y)$ ,  $\psi_1(y)$ ,  $\psi_2(y)$  and  $\psi_3(y)$  may for convenience be re-expressed in terms of  $\rho_i$  ( $i = 1$  to  $4$ ):

$$\rho_1(y)\xi(y) = \frac{\bar{\Lambda}}{2}[\psi_1(y) - 2(y-1)\psi_2(y) + 6\psi_3(y)] \quad (30)$$

$$\rho_2(y)\xi(y) = \frac{\bar{\Lambda}}{2}[\psi_1(y) - 2\psi_3(y)] \quad (31)$$

$$\rho_3(y)\xi(y) = \bar{\Lambda}\psi_2(y) \quad (32)$$

$$\rho_4(y)\xi(y) = -\frac{\bar{\Lambda}}{2}[(1+y)\xi_+(y) + (1-y)\xi(y)] \quad (33)$$

Thus including corrections from both the sources (current and state modifications) the formfactors  $\xi_i$  (defined in eqs. 11 and 12) could be written as

$$\xi_i = [\alpha_i + \gamma_i]\xi \quad (34)$$

where  $\gamma_i$  are given in terms of  $\rho_i(y)$ s by

$$\gamma_V = \frac{1}{2} \frac{\bar{\Lambda}}{m_s} + \frac{1}{m_s} \rho_2(y) \quad (35)$$

$$\gamma_{A_1} = \frac{1}{2} \frac{\bar{\Lambda}}{m_s} \frac{y-1}{y+1} + \frac{1}{m_s} \rho_2(y) \quad (36)$$

$$\gamma_{A_2} = \frac{1}{y+1} \frac{1}{m_s} [-\bar{\Lambda} + (y+1)\rho_3(y) - \rho_4(y)] \quad (37)$$

$$\gamma_{A_3} = \frac{1}{2} \frac{\bar{\Lambda}}{m_s} \frac{y-1}{y+1} + \frac{1}{m_s} (\rho_2(y) - \rho_3(y) - \frac{1}{y+1} \rho_4(y)) \quad (38)$$

Accordingly the decay width for  $B \rightarrow K^* \gamma$  including  $1/m_s$  correction is given by,

$$\begin{aligned} \Gamma(B \rightarrow K^* \gamma) = & \frac{G_F^2}{128\pi^4} \alpha |V_{tb}|^2 |V_{ts}|^2 |C_7(\mu)|^2 m_b^2 |\xi(y)|^2 \left(1 - \frac{M_{K^*}^2}{M_B}\right) M_{K^*} \\ & [(M_B - M_{K^*})^2 \{3(1 + 2\gamma_{A_1})(y + 1)^2 + (1 + 2\gamma_{A_3})(1 - y^2)\} \\ & + 2(M_B + M_{K^*})^2 (1 + 2\gamma_V)(y^2 - 1)] \end{aligned} \quad (39)$$

However, the form of the Isgur-Wise function (as also those which arise in the next-to-leading order) are model dependent and different models lead to a range of values for the the width being considered. Here we shall confine our attention to such models which provide estimates of the non-leading orders, for it is the purpose of this work to study this aspect of the problem.

### III. MODELS AND RESULTS

The form of the Isgur-Wise function, as also  $\rho_i$  ( $i=1$  to  $4$ ), shall next be obtained from the improved ISGW model [6] as well as from the BSW model [7].

#### 1. Improved Isgur-Scora-Grinstein-Wise Model

In the ISGW model [5], an S-wave  $Q\bar{d}$  meson  $X$  (here  $Q = b$  or  $s$  and  $X = B$  or  $K^*$ ) is represented by,

$$|X(\vec{p}_X, s_X)\rangle = \int d^3\vec{k} \sum_{s, \bar{s}} \phi_X(k) \chi_{s\bar{s}} |Q[\vec{p}_Q(\vec{p}_X, \vec{k}), s] \bar{d}[\vec{p}_d(\vec{p}_X, \vec{k}), r]\rangle \quad (40)$$

where  $\phi_X(k)$  is the wave function of the relevant  $Q\bar{d}$  S-wave state as a function of the relative momentum  $k$ ,  $\chi_{s\bar{s}}$  describes the coupling of the spins of the constituent quarks into the meson (pseudoscalar or vector, having polarisation vector  $\epsilon$ ). The momenta of the quarks,  $\vec{p}_Q$  and  $\vec{p}_d$ , are given by,

$$\vec{p}_Q = m_Q \vec{v}_X + \vec{k} \quad (41)$$

$$\vec{p}_d = m_d \vec{v}_X - \vec{k} \quad (42)$$

where  $\vec{v}_X$  is velocity of the meson and  $\vec{k}$  is the exchange momentum of the constituent quarks inside the meson.

The wave function  $\phi_X(k)$  is taken as the Fourier transform of the solution of the Schrödinger equation with a Hamiltonian

$$H_{ISGW} = -\frac{\nabla_Q^2}{2m_Q} - \frac{\nabla_d^2}{2m_d} - \frac{4\alpha_s}{3r} + a_1 r + a_2 \quad (43)$$

and using the variational method with oscillator wave-functions as trial functions for the required ground states viz.

$$\psi_{1S}(r) = \left(\frac{\beta_S}{\sqrt{\pi}}\right)^{3/2} \exp[-\beta_S^2 r^2/2] \quad (44)$$

and, evaluating the matrix element of the current, and going to the heavy quark limit ( $m_Q = M_X$  and  $\phi_B = \phi_{K^*}$ ), provides us with the ISGW estimates for the leading order form-factors, which, while in accordance with the heavy quark symmetry at the kinematic point  $y = 1$ , fails to conform at arbitrary values of  $y$  (as the same universal Isgur-Wise function do not emerge from all the form-factors). In order to rectify this deficiency Amundson [6] proposed an improved version of the ISGW model where the pseudoscalar (in our case  $B$ ) and vector (here  $K^*$ ) meson states are given by

$$|B(v)\rangle = \int d^3\vec{k} \sum_{r,s} \phi_B(k) \bar{u}_s(p_b(v, k), s) \gamma_5 v_d(p_d(v, k), r) \quad (45)$$

and

$$|K^*(v')\rangle = \int d^3\vec{k}' \sum_{r',s'} \phi_K^*(k') \bar{u}_s(p_s(v', k'), s') \not{\epsilon} v_d(p_d(v', k'), r') \quad (46)$$

The heavy quark limit is easily arrived at by replacing the meson wave-functions  $\phi_M$  by  $\phi_\infty$  (the limit of  $\phi_M$  as  $M_Q \rightarrow \infty$ ), which is the solution of the Hamiltonian,

$$H_0 = -\frac{\nabla_d^2}{2m_d} + V_{spinless} \quad (47)$$

where  $V_{spinless}$  is the part of the quark-antiquark binding potential inside the meson that does not depend on the spins of the constituent quarks. Thus the relevant matrix elements are

$$\begin{aligned}
\langle K^* | V^\mu | B \rangle &= - \int d^3k' d^3k \phi_\infty(k') \phi_\infty(k) \delta^3(\vec{p}_d - \vec{p}_d') \\
&\quad \sum_{rr', ss'} \bar{v}_d(p_d'(v', k'), r') \not{\epsilon} u_s(p_s(v', k'), s') \\
&\quad \bar{u}_s(p_s(v', k'), s') \gamma^\mu u_b(p_b(v, k), s) \\
&\quad \bar{u}_b(p_b(v, k), s) \gamma_5 v_d(p_d(v, k), r) \\
&= -Tr[\not{\epsilon} \frac{1 + \not{\psi}'}{2} \gamma^\mu \frac{1 + \not{\psi}}{2} \gamma_5] \\
&\quad \int d^3k' d^3k \phi_\infty(k') \phi_\infty(k) \delta^3(\vec{p}_d - \vec{p}_d') \quad (48)
\end{aligned}$$

and

$$\begin{aligned}
\langle K^* | A^\mu | B \rangle &= - \int d^3k' d^3k \phi_\infty(k') \phi_\infty(k) \delta^3(\vec{p}_d - \vec{p}_d') \\
&\quad \sum_{rr', ss'} \bar{v}_d(p_d'(v', k'), r') \not{\epsilon} u_s(p_s(v', k'), s') \\
&\quad \bar{u}_s(p_s(v', k'), s') \gamma^\mu \gamma_5 u_b(p_b(v, k), s) \\
&\quad \bar{u}_b(p_b(v, k), s) \gamma_5 v_d(p_d(v, k), r) \\
&= -Tr[\not{\epsilon} \frac{1 + \not{\psi}'}{2} \gamma^\mu \gamma_5 \frac{1 + \not{\psi}}{2} \gamma_5] \\
&\quad \int d^3k' d^3k \phi_\infty(k') \phi_\infty(k) \delta^{(3)}(\vec{p}_d - \vec{p}_d') \quad (49)
\end{aligned}$$

From the above equations, the universal Isgur-Wise function is identified as,

$$\xi(y) = \int d^3k' d^3k \phi_\infty(k') \phi_\infty(k) \delta^{(3)}(\vec{p}_d - \vec{p}_d') \quad (50)$$

which after performing the integrations is given by,

$$\xi(y) = \exp[-\frac{m_d^2}{2\beta^2}(y - 1)] \quad (51)$$

The deficiency in the nonrelativistic model is compensated for by introducing a phenomenological parameter  $\kappa$  modifying the exponent in  $\xi$  replacing  $\beta$  by

$\kappa\beta$ . The value of  $\kappa$  is fixed to be  $0.6 \pm .06$  by fitting with the experimentally determined value of the “charge radius” ( $\rho = 0.93 \pm 0.10$ ) corresponding to the usual parametrization of the Isgur-Wise function  $\xi(y)$  as  $1 - \rho^2(y - 1) + \mathcal{O}[(y - 1)^2]$ .

Similarly, the current correction gives a contribution

$$- \frac{1}{2m_s} \text{Tr}[A_\alpha \not{\epsilon} \frac{1 + \not{\psi}'}{2} \gamma^\alpha \Gamma^\mu \frac{1 + \not{p}}{2} \gamma_5] \quad (52)$$

where

$$A_\alpha = \int d^3k' \int d^3k \phi_\infty(\vec{k}') \phi_\infty(\vec{k}) l_\alpha(k') \delta^{(3)}(\vec{p}_d - \vec{p}_d') \quad (53)$$

Comparing the results of the improved ISGW model ( eq.(52) and eq.(53) ) with the general expression for the current correction (eq.(19) - eq.(22)), we get the two unknown parameters, arising from the current correction of order  $\frac{1}{m_s}$ , namely

$$\bar{\Lambda} = m_d \quad (54)$$

and

$$\xi_+(y) = 0 \quad (55)$$

It may be observed that in this model the term of the form  $(1 - y)\xi\gamma_\alpha$ , expected in general, is absent; this is a limitation of this nonrelativistic model.

Now the correction to the wave function from the heavy quark limit  $\phi_\infty$  has to be taken into account. Incorporating this correction, the wavefunction has the form,

$$\phi_{K^*} = \phi_\infty + \phi^1 \quad (56)$$

where  $\phi^1$  is the correction to be estimated in the first order of perturbation. This introduces a contribution to the matrix element of the vector and axial vector current involving

$$\int d^3k' \int d^3k \phi^1(\vec{k}') \phi_\infty(\vec{k}) \delta^{(3)}(\vec{p}_d - \vec{p}_d') \quad (57)$$

To calculate these corrections [12], one takes the kinetic energy term ( $\mathcal{H}_{KE}$ ) of heavy quark  $-\frac{\nabla_d^2}{2m_s}$  and the spin-spin interaction ( $\mathcal{H}_{SS}$ ) term  $\frac{gS_d \cdot S_s}{2m_d m_s} \delta^{(3)}(\vec{r})$  as perturbation, to obtain

$$\phi_{K^*} = \phi_\infty + \phi_{KE}^1 + \phi_{SS}^1 \quad (58)$$

where, denoting the  $n$ th wavefunction of the complete set of basis states (the eigenstates of the unperturbed Hamiltonian) by  $\phi_n$ ,

$$\begin{aligned} \phi_{KE}^1 &= \sum_{n \neq \infty} \frac{\phi_n(r)}{E_n - E_\infty} \int d^3 r' \phi_n^*(r') \mathcal{H}_{KE} \phi_\infty(r') \\ &\approx -[\frac{3}{2}]^{1/2} \frac{\beta^2}{2m_s(E_{2S} - E_{1S})} \psi_{2S}(r) \end{aligned} \quad (59)$$

retaining only the 2S-state expected to give a significant contribution with

$$\psi_{2S}(r) = \sqrt{\frac{2}{3}} \left( \frac{\beta_S}{\sqrt{\pi}} \right)^{3/2} \left( \beta^2 r^2 - \frac{3}{2} \right) \exp[-\beta_S^2 r^2 / 2] \quad (60)$$

Comparing equation(57) with equation(27) and inserting the model wavefunctions from from eq.(44) and eq.(59), we obtain

$$\begin{aligned} \psi_1 &= \frac{2m_s}{\Lambda} \int d^3 k' \int d^3 k \phi_{KE}^1(k') \phi_\infty(k) \delta^{(3)}(\vec{p}_d - \vec{p}_d') \\ &= \frac{m_s}{2(E_{2S} - E_{1S})} (y - 1) \xi(y) \end{aligned} \quad (61)$$

The energy denominator

$$E_{2S} - E_{1S} = \frac{\beta_S^2}{m_s} + \frac{a_1}{\beta_S \sqrt{\pi}} + \frac{4\alpha_s \beta_S}{9\sqrt{\pi}} \quad (62)$$

has the value 829 MeV with  $\beta_S$  and  $a_1$  taken to [ref] be 0.42 and 0.18 GeV<sup>2</sup> respectively. The observed [13] mass splitting of the 1S and 2S resonant states of  $K^*$  is about 818 MeV. The discrepancy is less than 1.5% which is acceptable considering the non-relativistic nature of the model.

Similarly, for the hyperfine interaction term ,

$$\begin{aligned}\phi_{SS}^1 &= \langle S_d \cdot S_s \rangle \sum_{n \neq \infty} \frac{\phi_n(r)}{E_n - E_\infty} \int d^3 r' \phi_n^*(r') \frac{g \delta(r')}{2m_d m_s} \phi_\infty(r') \\ &\approx \frac{1}{4} \frac{g}{2m_d m_s (E_{2S} - E_{1S})} \psi^{2S*}(0) \psi^{1S}(0) \psi^{2S}(r)\end{aligned}\quad (63)$$

with the dominant contribution coming from the first radial excitation  $\psi^{2S}$ .

In this model, as the variational method is used for the unperturbed wavefunction, it is not very reliable for the estimation of the value of the wavefunction at the origin which, however, can be estimated from available experimental data as

$$M_{K^*} - M_K = \frac{g}{2m_d m_s} |\psi^{1S}(0)|^2 \quad (64)$$

For potential models noting that

$$\frac{d^2}{dr^2} V(r) \geq 0 \Rightarrow |\psi^{2S}(0)|^2 \geq |\psi^{1S}(0)|^2 \quad (65)$$

we may use the fact that for heavy-light systems, the linear part of the potential is dominant for which  $\frac{d^2}{dr^2} V(r) \approx 0$  we are led to the interesting result

$$\psi^{2S}(0) \approx \psi^{1S}(0) \quad (66)$$

Accordingly we can rewrite  $\phi_{SS}^1$  as,

$$\frac{1}{4} \frac{M_{K^*} - M_K}{E_{2S} - E_{1S}} \psi^{2S}(r) \quad (67)$$

and again comparing eq. (57) with this form of  $\phi_{SS}^1$  and eq.(28) and inserting the relevant wave-functions, we have

$$\begin{aligned}\psi_3(y) &= -\frac{m_s}{\Lambda} \int d^3 k' d^3 k \phi_{SS}^1(k') \phi_\infty(k) \delta^3(\vec{p}_d - \vec{p}_d') \\ &= \frac{m_s m_d (M_{K^*} - M_K)}{4\beta^2 \sqrt{6} (E_{2S} - E_{1S})} (y-1) \xi(y)\end{aligned}\quad (68)$$

Thus the flavor and spin symmetry breaking  $\psi_1$  and  $\psi_3$  can be calculated. But in this model, there is no symmetry breaking term which can give a non-vanishing  $\psi_2$  (here  $\psi_2(y) = 0$ ). Accordingly, implementing the above results the decay width turns out to be  $1.1 \times 10^{-17} \text{ GeV}$  in the heavy quark limit of b and s quark with  $\beta_S = 0.42$ . This, however, is increased to  $1.8 \times 10^{-17} \text{ GeV}$  when  $1/m_s$  corrections are taken into account.

## 2. Bauer-Stech-Wirbel Model

The BSW model begins with the definition of the hadronic form-factors through

$$\begin{aligned} \langle K^* | j_\mu | B \rangle &= \frac{2}{M_B + M_{K^*}} \epsilon_{\mu\nu\rho\sigma} \epsilon^{*\nu} p^\rho p'^\sigma V(q^2) \\ &+ i \left\{ \epsilon_\mu^* (M_B + M_{K^*}) A_1(q^2) - \frac{\epsilon^* \cdot q}{M_B + M_{K^*}} (p + p')_\mu A_2(q^2) \right. \\ &\left. - \frac{\epsilon^* \cdot q}{q^2} 2M_{K^*} q_\mu A_3(q^2) \right\} + i \frac{\epsilon^* \cdot q}{q^2} 2M_{K^*} q_\mu A_0(q^2) \end{aligned} \quad (69)$$

where  $A_3(q^2)$  is simply an abbreviation for

$$A_3(q^2) = \frac{M_B + M_{K^*}}{2M_{K^*}} A_1(q^2) - \frac{M_B - M_{K^*}}{2M_{K^*}} A_2(q^2) \quad (70)$$

and in order to cancel the artificially introduced singularity at  $q^2 = 0$  we must have  $A_3(q^2 = 0) = A_0(q^2 = 0)$ . One goes on to relate these form-factors at maximum recoil (which is all that is needed here because for the real photon we have  $q^2 = 0$ ) to the overlap integral of the initial and final mesonic wave-functions in the infinite momentum frame, which depends on the fraction ( $x$ ) of the longitudinal momentum carried by the heavy quark and its square-averaged transverse momentum ( $\omega^2 = \langle p_T^2 \rangle$ ), and thus

$$\begin{aligned} V(q^2 = 0) &= \frac{m_b - m_s}{M_B - M_{K^*}} \langle K^* | \frac{1}{x} | B \rangle \\ &= \frac{m_b - m_s}{M_B - M_{K^*}} \int_0^1 dx \phi_{K^*}(x) \frac{1}{x} \phi_B(x) \end{aligned} \quad (71)$$

$$\begin{aligned}
A_1(q^2 = 0) &= \frac{m_b + m_s}{M_B + M_{K^*}} \langle K^* | \frac{1}{x} | B \rangle \\
&= \frac{m_b + m_s}{M_B + M_{K^*}} \int_0^1 dx \phi_{K^*}(x) \frac{1}{x} \phi_B(x)
\end{aligned} \tag{72}$$

$$\begin{aligned}
A_3(q^2 = 0) &= \langle K^* | B \rangle \\
&= \int_0^1 dx \phi_{K^*}(x) \phi_B(x)
\end{aligned} \tag{73}$$

Here  $\phi_{K^*}(x)$  and  $\phi_B(x)$  are taken generically from the ground state solution of a relativistic scalar harmonic oscillator potential, i.e.

$$\phi_M(x) = N_M \sqrt{x(1-x)} \exp \left[ -\frac{M^2}{2\omega^2} \left( 1 - x - \frac{\alpha}{M} \right)^2 \right] \tag{74}$$

where  $M = M_B$  or  $M_{K^*}$ . Taking the heavy quark limit one arrives at the Isgur-Wise function in the leading approximation, to wit,

$$\xi_{BSW}(y) = \frac{2}{y+1} \exp \left[ -\frac{2m_d^2}{\omega^2} \left( \frac{y-1}{y+1} \right) \right] \tag{75}$$

and the functions appearing in the next to leading order ( $\frac{1}{m_s}$  corections) are given by

$$\rho_1^{BSW}(y) = \frac{\Delta(y)}{y+1} \tag{76}$$

$$\rho_2^{BSW}(y) = -\frac{y - \sqrt{y^2 - 1}}{y+1} \Delta(y) \tag{77}$$

$$\rho_3^{BSW}(y) = \left( 1 - \sqrt{\frac{y-1}{y+1}} \right) \frac{\Delta(y)}{y-1} \tag{78}$$

$$\rho_4^{BSW}(y) = 0 \tag{79}$$

where

$$\Delta(y) = \frac{y-1}{2y} \alpha + \frac{\omega}{2} \left[ I\left(\frac{\alpha}{\omega}\right) - \sqrt{\frac{y+1}{2y}} I\left(\sqrt{\frac{y+1}{2y}} \frac{\alpha}{\omega}\right) \right] \tag{80}$$

$\alpha$  being the difference between the values of the masses of the meson and that of the heavy quark contained therein and  $\bar{\Lambda}$  as defined in eq.(22) is given in this model by

$$\alpha \left[ 1 + \frac{\omega}{2\alpha} I\left(\frac{\alpha}{\omega}\right) \right] \tag{81}$$

In all these expressions the function  $I$ , related to the error integral, is given by

$$I(x) \equiv \frac{\int_{-x}^{\infty} dz \exp(-z^2)}{\int_{-x}^{\infty} dz (z+x) \exp(-z^2)} \quad (82)$$

In this BSW model, one takes the value[ref]  $\omega = 400\text{MeV}$  irrespective of the flavor involved, though of course the value of the parameter  $\alpha$ , which is related to the mass difference between the heavy meson and the heavy quark would depend, in general, on the meson being considered and the flavor occuring therein. However, in the heavy quark limit (for both b and s) one should take this parameter to be flavor independent and we adopt the value[ref]  $\alpha = 280 \text{ MeV}$ . With these values of the parameters, the decay width for  $B \rightarrow K^* \gamma$  (both b and s taken as heavy) comes out to be  $2.2 \times 10^{-17} \text{ GeV}$ , wherein after incorporating the  $1/m_s$  correction one arrives at the value  $3.8 \times 10^{-17} \text{ GeV}$ .

### 3. A general treatment assuming only the b-quark as heavy

Lastly, we consider the most reliable assumption, where only the bottom quark mass is taken to be infinity. The hadronic matrix element of tensor-type current relevant for  $B \rightarrow K^* \gamma$  may be represented as [14],

$$\begin{aligned} \langle K^*(p', \epsilon) | \bar{s} \sigma_{\mu\nu} b | B(p) \rangle &= g_+ \epsilon_{\mu\nu\lambda\sigma} \epsilon^{*\lambda} (p + p')^\sigma + g_- \epsilon_{\mu\nu\lambda\sigma} \epsilon^{*\lambda} (p - p')^\sigma \\ &+ h \epsilon_{\mu\nu\lambda\sigma} (p + p')^\lambda (p - p')^\sigma (\epsilon^* \cdot p) \end{aligned} \quad (83)$$

using the relation  $\sigma^{\mu\nu} = \frac{i}{2} \epsilon^{\mu\nu\lambda\sigma} \sigma_{\lambda\sigma} \gamma_5$ , in the preceeding equation, we may write

$$\begin{aligned} \langle K^*(p', \epsilon) | \bar{s} \sigma_{\mu\nu} \gamma_5 b | B(p) \rangle &= i g_+ [\epsilon_\nu^*(p + p')_\mu - \epsilon_\mu^*(p + p')_\nu] \\ &+ i g_- [\epsilon_\nu^*(p - p')_\mu - \epsilon_\mu^*(p - p')_\nu] \end{aligned}$$

$$\begin{aligned}
& + \quad ih[(p + p')_\nu(p - p')_\mu \\
& - (p - p')_\nu(p + p')_\mu](\epsilon^* \cdot p)
\end{aligned} \tag{84}$$

In the heavy quark symmetry limit, as  $m_b \rightarrow \infty$

$$\not{p}b = b$$

which in the rest frame of the B meson becomes

$$\gamma_0 b = b \tag{85}$$

Using the above equation the tensor form factors  $g_+$ ,  $g_-$  and  $h$  get related to the vector and axial vector type form factors  $V$ ,  $A_1$ ,  $A_2$  and  $A_3$  through

$$\langle K^*(p', \epsilon) | \bar{s} \sigma_{0i} b | B(0) \rangle = -i \langle K^*(p', \epsilon) | \bar{s} \gamma_i b | B(0) \rangle \tag{86}$$

$$\langle K^*(p', \epsilon) | \bar{s} \sigma_{0i} \gamma_5 b | B(0) \rangle = i \langle K^*(p', \epsilon) | \bar{s} \gamma_i \gamma_5 b | B(0) \rangle \tag{87}$$

The decay width for  $B \rightarrow K^* \gamma$  takes the form,

$$\begin{aligned}
\Gamma = & \frac{5}{4} \frac{G_F^2}{128\pi^4 M_B^5} \alpha |V_{tb}|^2 |V_{ts}|^2 |C_7(\mu)|^2 m_b^2 \\
& (M_B^2 - M_{K^*}^2)^3 \{ (M_{K^*} + M_B) A_1(0) + (M_B - M_{K^*}) V(0) \}^2
\end{aligned} \tag{88}$$

$A_1(0)$  and  $V(0)$  may then be obtained from the BSW model [15] and gives the decay width of magnitude  $2.4 \times 10^{-17} \text{ GeV}$ . The ISGW model is not suitable for providing estimates in this approach due to its nonrelativistic nature.

To calculate  $R(B \rightarrow K^* \gamma)$  (the ratio of the decay width of  $B \rightarrow K^* \gamma$  to that of the quark level process  $b \rightarrow s \gamma$  [8, 9] ( $B \rightarrow X_s \gamma$ )) one has to consider the decay width of  $b \rightarrow s \gamma$ , expressed as,

$$\Gamma(b \rightarrow s \gamma) = \frac{G_F^2 \alpha}{32\pi^4} |V_{tb}|^2 |V_{ts}|^2 |C_7(\mu)|^2 m_b^5 \tag{89}$$

The numerical value for the decay width of this quark-level process is  $1.3 \times 10^{-16} \text{GeV}$ .

The hadronisation ratio  $R = \Gamma(B \rightarrow K^* \gamma) / \Gamma(b \rightarrow s \gamma)$  calculated in all these approaches is shown in Table 1.

## IV. SUMMARY AND CONCLUSION

We have shown the results for hadronization fraction  $R = \Gamma(B \rightarrow K^* \gamma) / \Gamma(b \rightarrow s \gamma)$ , calculated in the frame work of Heavy Quark Effective Theory. the required formfactors are obtained from the improved ISGW as well as from the BSW model, for both the cases, results are shown first for a heavy s quark ( $m_s \sim \infty$ ) and then implementing the  $1/m_s$  correction. It may be seen that in the former case  $1/m_s$  correction is as much as 65% while in the latter case it is even larger (71%). Comparing with the experimentally observed value of  $R = 0.19 \pm 0.09$  [16], we see that for the the BSW model the calculated value of R overshoots from 0.17( both b and s heavy) to 0.30 (incorporating  $1/m_s$  corrections). The fact that taking only b to be heavy and keeping  $m_s$  at its physical value resulting in  $R = 0.18$  suggest that the higher order  $1/m_s$  corrections contribute with opposite sign. In any case, the results strongly indicate that the approach where the s quark is taken to be heavy is not only model-dependent but is also highly unreliable as the perturbation expansion in power of  $1/m_s$  does not seem to converge well and alternates in sign.

## References

- [1] ParticleData Group: Phys. Rev. **D50** (1994) 1173 (1994).
- [2] S. Bertolini, F. Borzumati, A. Masiero: Phys. Rev. Lett. **59** (1987) 180;  
N.G. Deshpande *et al.*: Phys. Rev. Lett. **59** (1987) 183
- [3] M. Neubert: Phys. Rep. **C245** (1994) 259
- [4] A. Ali, T. Mannel: Phys. Lett. **B264** (1991) 447; **B274** (1992) 526(E)
- [5] N. Isgur, D. Scora, B. Grinstein, M. Wise: Phys. Rev. **D39** (1989) 799
- [6] J. F. Amundson: Phys. Rev. **D49** (1994) 373 and references therein;
- [7] M. Neubert, V. Rieckert: Nucl. Phys. **B382** (1992) 97 and references therein;
- [8] B. Grinstein, R. Springer, M. B. Wise: Nucl. Phys. **B339** (1990) 269
- [9] R. Grigjanis *et al.*: Phys. Rep. **228** (1993) 93
- [10] M. Neubert *et al.* in *Heavy Flavours*, ed. A. J. Buras and M. Lindner (World Scientific, 1992).
- [11] M. E. Luke: Phys. Lett. **B252** (1990) 447
- [12] C. Dib, F. Vera: Phys. Rev. **D47** (1993) 3938
- [13] M. R. Ahmady, D. Liu: Phys. Lett. **B324** (1994) 231
- [14] N. Isgur, M. Wise: Phys. Rev. **D42** (1990) 2388; G. Burdman, J. F. Donoghue: Phys. Lett. **B270** (1991) 55

- [15] M. Wirbel, B. Stech, M. Bauer: Z Phys. **C29** (1985) 637; *ibid.*, **C34** (1987) 103
- [16] T. E. Browder, K. Honscheid: Prog. Part. Nucl. Phys. **35** (1995) 81

TABLE I

| Model         | Condition                                       | Decay width<br>in GeV  | Branching fraction<br>$= \frac{\Gamma(B \rightarrow K^* \gamma)}{\Gamma(B \rightarrow \text{all})}$ | $R(B \rightarrow K^* \gamma)$<br>$= \frac{\Gamma(B \rightarrow K^* \gamma)}{\Gamma(b \rightarrow s \gamma)}$ |
|---------------|---|------------------------|---|--|
| Improved ISGW | b & s as heavy<br>(without $1/m_s$ corrections) | $1.10 \times 10^{-17}$ | $2.51 \times 10^{-5}$   | .09  |
| Improved ISGW | b & s as heavy<br>(with $1/m_s$ corrections)    | $1.82 \times 10^{-17}$ | $4.15 \times 10^{-5}$   | .14  |
| BSW           | b & s as heavy<br>(without $1/m_s$ corrections) | $2.23 \times 10^{-17}$ | $5.08 \times 10^{-5}$   | .17  |
| BSW           | b & s as heavy<br>(with $1/m_s$ corrections)    | $3.82 \times 10^{-17}$ | $8.70 \times 10^{-5}$   | .30  |
| BSW           | Only b as heavy                                 | $2.35 \times 10^{-17}$ | $5.35 \times 10^{-5}$   | .18  |